9.2 Use Properties of Matrices

Before You performed translations using vectors.

Now You will perform translations using matrix operations.

Why So you can calculate the total cost of art supplies, as in Ex. 36.

Key Vocabulary
- **matrix**
- **element**
- **dimensions**

A **matrix** is a rectangular arrangement of numbers in rows and columns. (The plural of matrix is *matrices*.) Each number in a matrix is called an **element**.

column

$$\text{row}\quad\begin{bmatrix} 5 & 4 & 4 & 9 \\ -3 & 5 & 2 & 6 \\ 3 & -7 & 8 & 7 \end{bmatrix}$$

← **The element in the second row and third column is 2.**

READ VOCABULARY
An element of a matrix may also be called an *entry*.

The **dimensions** of a matrix are the numbers of rows and columns. The matrix above has three rows and four columns, so the dimensions of the matrix are 3×4 (read "3 by 4").

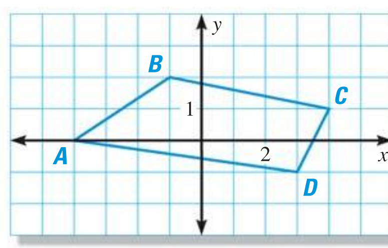
You can represent a figure in the coordinate plane using a matrix with two rows. The first row has the x-coordinate(s) of the vertices. The second row has the corresponding y-coordinate(s). Each column represents a vertex, so the number of columns depends on the number of vertices of the figure.

EXAMPLE 1 Represent figures using matrices

Write a matrix to represent the point or polygon.

a. Point A

b. Quadrilateral $ABCD$



Solution

a. Point matrix for A

$$\begin{bmatrix} -4 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinate} \\ \leftarrow y\text{-coordinate} \end{matrix}$$

b. Polygon matrix for $ABCD$

$$\begin{matrix} & A & B & C & D \\ & \end{matrix}$$
$$\begin{bmatrix} -4 & -1 & 4 & 3 \\ 0 & 2 & 1 & -1 \end{bmatrix} \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix}$$

AVOID ERRORS
The columns in a polygon matrix follow the consecutive order of the vertices of the polygon.

✓ **GUIDED PRACTICE** for Example 1

1. Write a matrix to represent $\triangle ABC$ with vertices $A(3, 5)$, $B(6, 7)$ and $C(7, 3)$.

2. How many rows and columns are in a matrix for a hexagon?

ADDING AND SUBTRACTING To add or subtract matrices, you add or subtract corresponding elements. The matrices must have the same dimensions.

EXAMPLE 2 Add and subtract matrices

a. $\begin{bmatrix} 5 & -3 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 5+1 & -3+2 \\ 6+3 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 9 & -10 \end{bmatrix}$

b. $\begin{bmatrix} 6 & 8 & 5 \\ 4 & 9 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -7 & 0 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 6-1 & 8-(-7) & 5-0 \\ 4-4 & 9-(-2) & -1-3 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 5 \\ 0 & 11 & -4 \end{bmatrix}$

TRANSLATIONS You can use matrix addition to represent a translation in the coordinate plane. The image matrix for a translation is the sum of the translation matrix and the matrix that represents the preimage.

EXAMPLE 3 Represent a translation using matrices

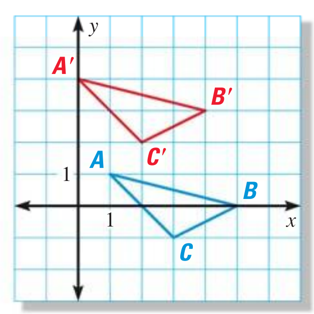
The matrix $\begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & -1 \end{bmatrix}$ represents $\triangle ABC$. Find the image matrix that represents the translation of $\triangle ABC$ 1 unit left and 3 units up. Then graph $\triangle ABC$ and its image.

Solution

The translation matrix is $\begin{bmatrix} -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix}$.

Add this to the polygon matrix for the preimage to find the image matrix.

$$\underset{\text{Translation matrix}}{\begin{bmatrix} -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix}} + \underset{\text{Polygon matrix}}{\overset{\quad A \quad B \quad C}{\begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & -1 \end{bmatrix}}} = \underset{\text{Image matrix}}{\overset{\quad A' \quad B' \quad C'}{\begin{bmatrix} 0 & 4 & 2 \\ 4 & 3 & 2 \end{bmatrix}}}$$



AVOID ERRORS
In order to add two matrices, they must have the same dimensions, so the translation matrix here must have three columns like the polygon matrix.

GUIDED PRACTICE for Examples 2 and 3

In Exercises 3 and 4, add or subtract.

3. $\begin{bmatrix} -3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & -5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -4 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$

5. The matrix $\begin{bmatrix} 1 & 2 & 6 & 7 \\ 2 & -1 & 1 & 3 \end{bmatrix}$ represents quadrilateral *JKLM*. Write the translation matrix and the image matrix that represents the translation of *JKLM* 4 units right and 2 units down. Then graph *JKLM* and its image.

MULTIPLYING MATRICES The product of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix.

USE NOTATION
Recall that the dimensions of a matrix are always written as rows $\times$ columns.

$$A \quad \cdot \quad B \quad = \quad AB$$

$$(m \text{ by } n) \quad \cdot \quad (n \text{ by } p) \quad = \quad (m \text{ by } p)$$

equal — dimensions of *AB*

You will use matrix multiplication in later lessons to represent transformations.

EXAMPLE 4 Multiply matrices

Multiply $\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix}$.

Solution

The matrices are both 2×2, so their product is defined. Use the following steps to find the elements of the product matrix.

STEP 1 **Multiply** the numbers in the first row of the first matrix by the numbers in the first column of the second matrix. Put the result in the first row, first column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & ? \\ ? & ? \end{bmatrix}$$

STEP 2 **Multiply** the numbers in the first row of the first matrix by the numbers in the second column of the second matrix. Put the result in the first row, second column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ ? & ? \end{bmatrix}$$

STEP 3 **Multiply** the numbers in the second row of the first matrix by the numbers in the first column of the second matrix. Put the result in the second row, first column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & ? \end{bmatrix}$$

STEP 4 **Multiply** the numbers in the second row of the first matrix by the numbers in the second column of the second matrix. Put the result in the second row, second column of the product matrix.

$$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & 4(-3) + 5(8) \end{bmatrix}$$

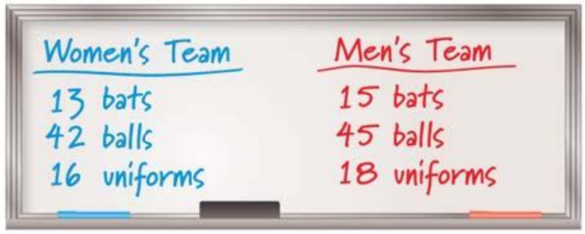
STEP 5 **Simplify** the product matrix.

$$\begin{bmatrix} 1(2) + 0(-1) & 1(-3) + 0(8) \\ 4(2) + 5(-1) & 4(-3) + 5(8) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 28 \end{bmatrix}$$

Animated Geometry at classzone.com

EXAMPLE 5 Solve a real-world problem

SOFTBALL Two softball teams submit equipment lists for the season. A bat costs \$20, a ball costs \$5, and a uniform costs \$40. Use matrix multiplication to find the total cost of equipment for each team.



Solution

ANOTHER WAY
You could solve this problem arithmetically, multiplying the number of bats by the price of bats, and so on, then adding the costs for each team.

First, write the equipment lists and the costs per item in matrix form. You will use matrix multiplication, so you need to set up the matrices so that the number of columns of the equipment matrix matches the number of rows of the cost per item matrix.

EQUIPMENT · **COST** = **TOTAL COST**

$$\begin{array}{c} \\ \text{Women} \\ \text{Men} \end{array}\begin{array}{c} \begin{array}{ccc}\text{Bats} & \text{Balls} & \text{Uniforms}\end{array} \\ \begin{bmatrix} 13 & 42 & 16 \\ 15 & 45 & 18 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ \text{Bats} \\ \text{Balls} \\ \text{Uniforms} \end{array}\begin{array}{c} \text{Dollars} \\ \begin{bmatrix} 20 \\ 5 \\ 40 \end{bmatrix} \end{array} = \begin{array}{c} \\ \text{Women} \\ \text{Men} \end{array}\begin{array}{c} \text{Dollars} \\ \begin{bmatrix} ? \\ ? \end{bmatrix} \end{array}$$

You can find the total cost of equipment for each team by multiplying the equipment matrix by the cost per item matrix. The equipment matrix is 2×3 and the cost per item matrix is 3×1, so their product is a 2×1 matrix.

$$\begin{bmatrix} 13 & 42 & 16 \\ 15 & 45 & 18 \end{bmatrix}\begin{bmatrix} 20 \\ 5 \\ 40 \end{bmatrix} = \begin{bmatrix} 13(20) + 42(5) + 16(40) \\ 15(20) + 45(5) + 18(40) \end{bmatrix} = \begin{bmatrix} 1110 \\ 1245 \end{bmatrix}$$

▶ The total cost of equipment for the women's team is \$1110, and the total cost for the men's team is \$1245.

✓ GUIDED PRACTICE for Examples 4 and 5

Use the matrices below. Is the product defined? *Explain*.

$$A = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 6.7 & 0 \\ -9.3 & 5.2 \end{bmatrix}$$

6. AB **7.** BA **8.** AC

Multiply.

9. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -4 & 7 \end{bmatrix}$ **10.** $\begin{bmatrix} 5 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ **11.** $\begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}$

12. WHAT IF? In Example 5, find the total cost if a bat costs \$25, a ball costs \$4, and a uniform costs \$35.

9.2 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 13, 19, and 31

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 17, 24, 25, and 35

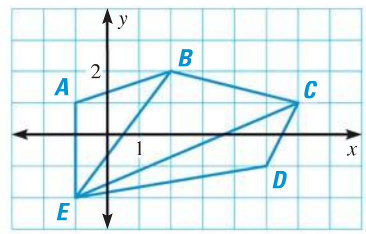
SKILL PRACTICE

1. **VOCABULARY** Copy and complete: To find the sum of two matrices, add corresponding __?__.

2. ★ **WRITING** How can you determine whether two matrices can be added? How can you determine whether two matrices can be multiplied?

EXAMPLE 1 on p. 580 for Exs. 3–6

USING A DIAGRAM **Use the diagram to write a matrix to represent the given polygon.**



3. $\triangle EBC$

4. $\triangle ECD$

5. Quadrilateral *BCDE*

6. Pentagon *ABCDE*

EXAMPLE 2 on p. 581 for Exs. 7–12

MATRIX OPERATIONS **Add or subtract.**

7. $\begin{bmatrix} 3 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -12 & 5 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 8 \end{bmatrix}$

9. $\begin{bmatrix} 9 & 8 \\ -2 & 3 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ 2 & -3 \\ -5 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 4.6 & 8.1 \end{bmatrix} - \begin{bmatrix} 3.8 & -2.1 \end{bmatrix}$

11. $\begin{bmatrix} -5 & 6 \\ -8 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 10 \\ 4 & -7 \end{bmatrix}$

12. $\begin{bmatrix} 1.2 & 6 \\ 5.3 & 1.1 \end{bmatrix} - \begin{bmatrix} 2.5 & -3.3 \\ 7 & 4 \end{bmatrix}$

EXAMPLE 3 on p. 581 for Exs. 13–17

TRANSLATIONS **Find the image matrix that represents the translation of the polygon. Then graph the polygon and its image.**

13. $\begin{matrix} A & B & C \end{matrix}$ $\begin{bmatrix} -2 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix}$; 4 units up

14. $\begin{matrix} F & G & H & J \end{matrix}$ $\begin{bmatrix} 2 & 5 & 8 & 5 \\ 2 & 3 & 1 & -1 \end{bmatrix}$; 2 units left and 3 units down

15. $\begin{matrix} L & M & N & P \end{matrix}$ $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -1 & 3 & 3 & -1 \end{bmatrix}$; 4 units right and 2 units up

16. $\begin{matrix} Q & R & S \end{matrix}$ $\begin{bmatrix} -5 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}$; 3 units right and 1 unit down

17. ★ **MULTIPLE CHOICE** The matrix that represents quadrilateral *ABCD* is $\begin{bmatrix} 3 & 8 & 9 & 7 \\ 3 & 7 & 3 & 1 \end{bmatrix}$. Which matrix represents the image of the quadrilateral after translating it 3 units right and 5 units up?

Ⓐ $\begin{bmatrix} 6 & 11 & 12 & 10 \\ 8 & 12 & 8 & 6 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 0 & 5 & 6 & 4 \\ 8 & 12 & 8 & 6 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 6 & 11 & 12 & 10 \\ -2 & 2 & -2 & -4 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 0 & 6 & 6 & 4 \\ -2 & 3 & -2 & -4 \end{bmatrix}$

EXAMPLE 4
on p. 582
for Exs. 18–26

MATRIX OPERATIONS **Multiply.**

18. $\begin{bmatrix} 5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

19. $\begin{bmatrix} 1.2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ -1.5 \end{bmatrix}$

20. $\begin{bmatrix} 6 & 7 \\ -5 & 8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 9 & -3 \end{bmatrix}$

21. $\begin{bmatrix} 0.4 & 6 \\ -6 & 2.3 \end{bmatrix}\begin{bmatrix} 5 & 8 \\ -1 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 4 & 8 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

23. $\begin{bmatrix} 9 & 1 & 2 \\ 8 & -1 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

24. ★ **MULTIPLE CHOICE** Which product is not defined?

Ⓐ $\begin{bmatrix} 1 & 7 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 6 \\ 15 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 3 & 20 \end{bmatrix}\begin{bmatrix} 9 \\ 30 \end{bmatrix}$ Ⓒ $\begin{bmatrix} 15 \\ -3 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix}$ Ⓓ $\begin{bmatrix} 30 \\ -7 \end{bmatrix}\begin{bmatrix} 5 & 5 \end{bmatrix}$

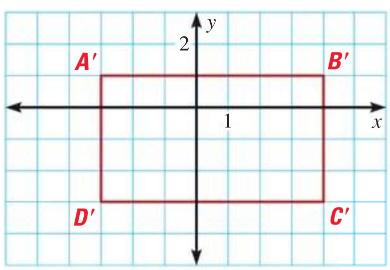
25. ★ **OPEN-ENDED MATH** Write two matrices that have a defined product. Then find the product.

26. **ERROR ANALYSIS** *Describe* and correct the error in the computation.

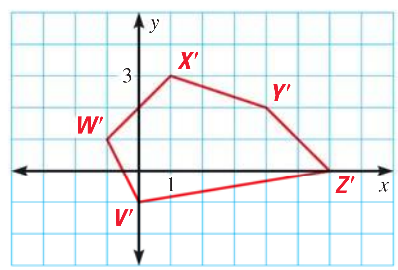
$$\begin{bmatrix} 9 & -2 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} 9(-6) & -2(12) \\ 4(3) & 10(-6) \end{bmatrix}$$

TRANSLATIONS **Use the described translation and the graph of the image to find the matrix that represents the preimage.**

27. 4 units right and 2 units down



28. 6 units left and 5 units up



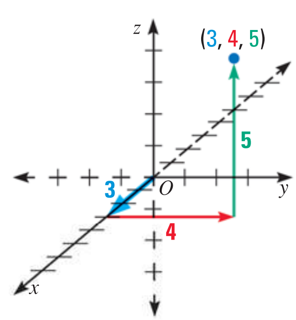
29. **MATRIX EQUATION** Use the description of a translation of a triangle to find the value of each variable. *Explain* your reasoning. What are the coordinates of the vertices of the image triangle?

$$\begin{bmatrix} 12 & 12 & w \\ -7 & v & -7 \end{bmatrix} + \begin{bmatrix} 9 & a & b \\ 6 & -2 & c \end{bmatrix} = \begin{bmatrix} m & 20 & -8 \\ n & -9 & 13 \end{bmatrix}$$

30. **CHALLENGE** A point in space has three coordinates (x, y, z), as shown at the right. From the origin, a point can be forward or back on the x-axis, left or right on the y-axis, and up or down on the z-axis.

a. You translate a point three units forward, four units right, and five units up. Write a translation matrix for the point.

b. You translate a figure that has five vertices. Write a translation matrix to move the figure five units back, ten units left, and six units down.



PROBLEM SOLVING

EXAMPLE 5
on p. 583
for Ex. 31

31. **COMPUTERS** Two computer labs submit equipment lists. A mouse costs \$10, a package of CDs costs \$32, and a keyboard costs \$15. Use matrix multiplication to find the total cost of equipment for each lab.

Lab 1	Lab 2
25 Mice	15 Mice
10 CDs	20 CDs
18 Keyboards	12 Keyboards

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32. **SWIMMING** Two swim teams submit equipment lists. The women's team needs 30 caps and 26 goggles. The men's team needs 15 caps and 25 goggles. A cap costs \$10 and goggles cost \$15.

   a. Use matrix addition to find the total number of caps and the total number of goggles for each team.

   b. Use matrix multiplication to find the total equipment cost for each team.

   c. Find the total cost for both teams.

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MATRIX PROPERTIES **In Exercises 33–35, use matrices *A*, *B*, and *C*.**

$$A = \begin{bmatrix} 5 & 1 \\ 10 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 4 \\ -5 & 1 \end{bmatrix}$$

33. **MULTI-STEP PROBLEM** Use the 2×2 matrices above to explore the Commutative Property of Multiplication.

   a. What does it mean that multiplication is *commutative*?

   b. Find and *compare* AB and BA.

   c. Based on part (b), make a conjecture about whether matrix multiplication is commutative.

34. **MULTI-STEP PROBLEM** Use the 2×2 matrices above to explore the Associative Property of Multiplication.

   a. What does it mean that multiplication is *associative*?

   b. Find and *compare* $A(BC)$ and $(AB)C$.

   c. Based on part (b), make a conjecture about whether matrix multiplication is associative.

35. ★ **SHORT RESPONSE** Find and *compare* $A(B + C)$ and $AB + AC$. Make a conjecture about matrices and the Distributive Property.

36. **ART** Two art classes are buying supplies. A brush is \$4 and a paint set is \$10. Each class has only \$225 to spend. Use matrix multiplication to find the maximum number of brushes Class A can buy and the maximum number of paint sets Class B can buy. *Explain*.

Class A	Class B
x brushes	18 brushes
12 paint sets	y paint sets

○ = WORKED-OUT SOLUTIONS on p. WS1   ★ = STANDARDIZED TEST PRACTICE

37. **CHALLENGE** The total United States production of corn was 8,967 million bushels in 2002, and 10,114 million bushels in 2003. The table shows the percents of the total grown by four states.

a. Use matrix multiplication to find the number of bushels (in millions) harvested in each state each year.

b. How many bushels (in millions) were harvested in these two years in Iowa?

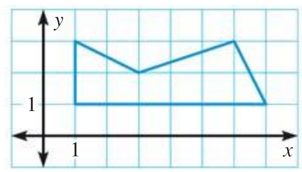
c. The price for a bushel of corn in Nebraska was \$2.32 in 2002, and \$2.45 in 2003. Use matrix multiplication to find the total value of corn harvested in Nebraska in these two years.

	2002	2003
Iowa	21.5%	18.6%
Illinois	16.4%	17.9%
Nebraska	10.5%	11.1%
Minnesota	11.7%	9.6%

MIXED REVIEW

PREVIEW
Prepare for Lesson 9.3 in Exs. 38–39.

Copy the figure and draw its image after the reflection. *(p. 272)*

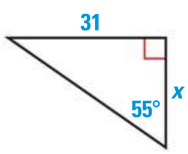


38. Reflect the figure in the x-axis.

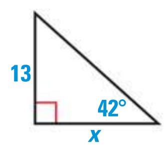
39. Reflect the figure in the y-axis.

Find the value of x to the nearest tenth. *(p. 466)*

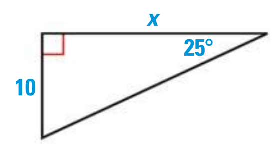
40.



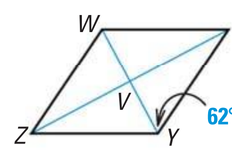
41.



42.



The diagonals of rhombus *WXYZ* intersect at *V*. Given that $m\angle XYW = 62°$, find the indicated measure. *(p. 533)*



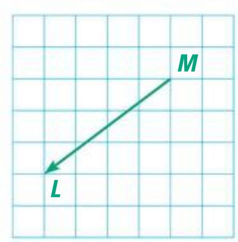
43. $m\angle ZYW =$ __?__

44. $m\angle WXY =$ __?__

45. $m\angle XVY =$ __?__

QUIZ *for Lessons 9.1–9.2*

1. In the diagram shown, name the vector and write its component form. *(p. 572)*



Use the translation $(x, y) \to (x + 3, y - 2)$. *(p. 572)*

2. What is the image of $(-1, 5)$?

3. What is the image of $(6, 3)$?

4. What is the preimage of $(-4, -1)$?

Add, subtract, or multiply. *(p. 580)*

5. $\begin{bmatrix} 5 & -3 \\ 8 & -2 \end{bmatrix} + \begin{bmatrix} -9 & 6 \\ 4 & -7 \end{bmatrix}$

6. $\begin{bmatrix} -6 & 1 \\ 3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 15 \\ -7 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 7 & -6 & 2 \\ 8 & 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -9 & 0 \\ 3 & -7 \end{bmatrix}$